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Abstract

In this article, we consider a certain subgroup of the IA-automorphism group of a free group, which we call the upper-triangular IA-automorphism group of a free group, and denote it by IA_n^+. We determine the images of the k-th Johnson homomorphism of IA_n^+ for any $k \geq 1$ and $n \geq 2$. By using this result, we give an affirmative answer to the Andreadakis conjecture restricted to IA_n^+. Namely, we show that the intersection of the Andreadakis-Johnson filtration and IA_n^+ coincides with the lower central series of IA_n^+.

In addition to this, we also consider the integral second (co)homology group of IA_n^+. In particular, we construct non-trivial second homology classes of IA_n^+ by observing its generators and relators, and show that the second cohomology group is not generated by cup products of the first cohomology groups.

In 1965, in his doctorial thesis, Andreadakis [1] introduced a descending filtration $\mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \cdots$ of the automorphism group $\mathrm{Aut}\, G$ of a group G. We call it the Andreadakis-Johnson filtration of $\mathrm{Aut}\, G$. One of the remarkable properties of the filtration $\{\mathcal{A}_G(k)\}$ is central. More precisely, he [1] showed that the commutator subgroup of $\mathcal{A}_G(k)$ and $\mathcal{A}_G(l)$ is contained in $\mathcal{A}_G(k+l)$ for any $k, l \geq 1$. Hence the graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for each $k \geq 1$ is an abelian group. In particular, it is known that if G is finitely generated, then so is $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for any $k \geq 1$. In general, the graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ are considered to be a sequence of approximations of $\mathrm{Aut}\, G$, and are one of powerful tools to study the group structure of $\mathrm{Aut}\, G$.

Let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. As is well known, one of the most basic and important groups is a free group in combinatorial group theory. Andreadakis [1] focused his interests on the

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study of the Andreadakis-Johnson filtration on $\mathrm{Aut}\,F_n$. For any group G, since the Andreadakis-Johnson filtration is central, the k-th subgroup $\mathcal{A}_G(k)$ contains that of the lower central series $\{\mathcal{A}'_G(k)\}$ of $\mathcal{A}_G(1)$ for each $k \geq 1$. Andreadakis [1] showed that $\mathcal{A}_{F_2}(k) = \mathcal{A}'_{F_2}(k)$ for any $k \geq 1$, and $\mathcal{A}_{F_3}(k) = \mathcal{A}'_{F_3}(k)$ for $k \leq 3$. In general, it is quite a difficult problem to determine whether $\mathcal{A}_G(k)$ coincides with $\mathcal{A}'_G(k)$ or not, even the case where $G = F_n$. It has been conjectured that $\mathcal{A}_{F_n}(k) = \mathcal{A}'_{F_n}(k)$ for any $n \geq 3$ and $k \geq 1$ by Andreadakis. Today, this conjecture is called the Andreadakis conjecture. For any $n \geq 2$, it is known that $\mathcal{A}_{F_n}(2) = \mathcal{A}'_{F_n}(2)$ due to Bachmuth [3], and that $\mathcal{A}'_{F_n}(3)$ has at most finite index in $\mathcal{A}_{F_n}(3)$ due to Pettet [29].

The reason why we call $\{\mathcal{A}_G(k)\}$ the Andreadakis-Johnson filtration is that it should be mentioned not only Andreadakis's original works for $\mathrm{Aut}\,F_n$ but also Johnson's results for mapping class groups of surfaces. The mapping class group of a compact oriented surface with one boundary component can be embedded into the automorphism group of a free group by classical works of Dehn and Nielsen in the 1910s and in early 1920s. Hence we can consider a descending filtration of the mapping class group by restricting the Andreadakis-Johnson filtration to it. The first subgroup of this filtration is called the Torelli subgroup of the mapping class group. In the 1980s, Johnson studied the group structure of the Torelli subgroup in a series of works [15], [16], [17] and [18]. In particular, he gave a finite set of generators of the Torelli group, and he constructed a homomorphism τ to determine the abelianization of it. Today, his homomorphism τ is called the first Johnson homomorphism, and it is generalized to Johnson homomorphisms of higher degrees. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the works of many authors including Morita [24], Hain [12] and others. The definition of the Johnson homomorphisms of the mapping class group can be easily generalized to those of $\mathrm{Aut}\,F_n$. To put it plainly, the Johnson homomorphisms are useful tools to study the graded quotients of the Andreadakis-Johnson filtration of $\mathrm{Aut}\,F_n$. (For details, see our survey papers [34] and [35].)

The first subgroup $\mathcal{A}_{F_n}(1)$ is called the IA-automorphism group of F_n, and usually denoted by IA_n. Bachmuth [2] called IA_n the IA-automorphism group since that consists of automorphisms which induce identity automorphisms on the abelianized group H of F_n. The letters I and A stands for "Identity" and "Automorphism" respectively. The subgroup IA_n reflects much richness and complexity of the structure of $\mathrm{Aut}\, F_n$, and plays important roles in various studies of $\mathrm{Aut}\, F_n$. In 1935, Magnus [21] showed that IA_n is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j{}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct i, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct i, j, $l \in \{1, 2, \ldots, n\}$ such that $j > l$. The group structure of IA_n is, however, less well understood. For instance, no presentation for IA_n is known for $n \geq 3$. Krstić and McCool [20] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not.

In this article, we consider a certain subgroup of IA_n. Let IA_n^+ be the subgroup of IA_n generated by K_{ij} for $1 \leq j < i \leq n$ and K_{ijl} for $1 \leq l < j < i \leq n$. The group IA_n^+ is an IA-automorphism group analogue of the group of the upper triangular matrices. We call IA_n^+ the upper-triangular IA-automorphism group of F_n. In our subsequent paper [36], we define the "upper-triangular" automorphism group A_n^+, which is a subgroup of $\mathrm{Aut}\, F_n$, and show that IA_n^+ coincides with the subgroup of A_n^+ consisting of automorphisms which act on H trivially. In the present paper, we give an affirmative answer to the Andreadakis conjecture restricted to IA_n^+. Namely, set $\mathcal{A}_{F_n}(k)^+ := \mathcal{A}_{F_n}(k) \cap \mathrm{IA}_n^+$ for each $k \geq 1$, and let $\{\mathcal{A}'_{F_n}(k)^+\}$ be the lower central series of IA_n^+. Then we show

Theorem 1. *For any $n \geq 2$ and $k \geq 1$, $\mathcal{A}_{F_n}(k)^+ = \mathcal{A}'_{F_n}(k)^+$.*

In order to prove this theorem, we use the Johnson homomorphisms

$$\tau_k'^{+} : \mathrm{gr}^k(\mathcal{A}_n'^{+}) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

of IA_n^+ where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual group of H. Frankly, they are defined by restricting those of $\mathrm{Aut}\, F_n$ to the graded quotients of the lower central series $\{\mathcal{A}'_{F_n}(k)^+\}$. In particular, we completely determine their images as follows.

Theorem 2. *For any $n \geq 2$ and $k \geq 1$, the image of $\tau_k'^+$ is a submodule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ generated by*

$$\{x_i^* \otimes [[\cdots[x_{j_1}, x_{j_2}], \ldots, x_{j_k}], x_i] \mid 1 \leq j_1, \ldots, j_k < i \leq n\}$$
$$\cup \{x_i^* \otimes [[\cdots[x_{j_1}, x_{j_2}], \ldots, x_{j_k}], x_{j_{k+1}}] \mid 1 \leq j_1, \ldots, j_{k+1} < i \leq n\}$$

where x_i^*s are the dual basis of H^*. Furthermore, we have

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_k'^+)) = \sum_{i=2}^{n} r_{i-1}(k) + \sum_{i=2}^{n} r_{i-1}(k+1).$$

Here $r_m(k)$ is the rank of the k-th graded quotient of the lower central series of F_m. More precisely, due to Witt [37], we have

$$r_m(k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$$

where μ is the Möbius function, and d runs over all positive divisors of k.

In [4], Bartholdi asserted that the "rational" Andreadakis conjecture is true by using the representation theory of the general linear group $\mathrm{GL}(n, \mathbf{Q})$. He also disprove the Andreadakis conjecture for $n = 3$ by giving brief descriptions of the procedure of a long computer calculation and its results. In general, to show $\mathcal{A}_{F_n}(k)/\mathcal{A}'_{F_n}(k) = 0$ is quite different thing to show $(\mathcal{A}_{F_n}(k)/\mathcal{A}'_{F_n}(k)) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. Bartholdi's representation theoretical proof cannot be applied to a proof of the Andreadakis conjecture for general $n \geq 4$, and hence it is still open problem. To the best of our knowledge, to attack the Andreadakis conjecture directly is too difficult and complicated to solve. Perhaps it might be worth considering to use the decomposition theorem with IA_n^+ on this problem. In general, "the upper-triangular type subgroup" has the much easier structure, and is useful to study the whole group. For example, the subgroup Λ_n of the general linear group $\mathrm{GL}(n, \mathbf{Z})$ consisting of all upper-triangular matrices has a very simple presentation. By using the presentation of Λ_n, and by using a kind of decomposition theorem for $\mathrm{GL}(n, \mathbf{Z})$ with Λ_n, Magnus [21] obtained finitely many generators of IA_n. If we consider to attack the Andreadakis conjecture for IA_n by constructing and using the decomposition theorem for IA_n with IA_n^+, our results on the paper seem to play important roles on this problem as a foothold.

Next, as applications of our results mentioned above, we consider to detect non-trivial homology classes in the integral second homology groups of IA_n^+. Let F be the free group generated by K_{ij} for $1 \leq j < i \leq n$ and K_{ijl} for $1 \leq l < j < i \leq n$, and $\pi : F \to \mathrm{IA}_n^+$ the natural surjection. We denote by R the kernel of π. Then by observing the homological five-term exact sequence of a group extension

$$1 \to R \to F \xrightarrow{\pi} \mathrm{IA}_n^+ \to 1,$$

we see $H_1(R, \mathbf{Z})_{\mathrm{IA}_n^+} \cong H_2(\mathrm{IA}_n^+, \mathbf{Z})$. For the lower central series $\Gamma_F(1) \supset \Gamma_F(2) \supset \cdots$ of F, set $R_k := R \cap \Gamma_F(k)$ and $\overline{R}_k := R/R_k$ for each $k \geq 1$. Then we have a surjective homomorphism

$$\psi_k : H_1(R, \mathbf{Z})_{\mathrm{IA}_n^+} \to H_1(R/R_{k+1}, \mathbf{Z})_{\mathrm{IA}_n^+}.$$

Then we can detect non-trivial second homology classes of IA_n^+ through ψ^k by studying the structure of each $H_1(R/R_{k+1}, \mathbf{Z})_{\mathrm{IA}_n^+}$. In the paper, we especially consider the case where $k = 2$ and 3. For $k = 2$, we easily see that $H_1(R/R_3, \mathbf{Z})_{\mathrm{IA}_n^+} = R/R_3$, and that R/R_3 is a free abelian group of rank $n(n^2-1)(n-2)^2(n^2+5n+9)/72$. Furthermore, by studying a group structure of $H_1(R/R_4, \mathbf{Z})_{\mathrm{IA}_n^+}$, we obtain $H_2(\mathrm{IA}_n^+, \mathbf{Z}) \ncong R/R_3$, and show

Theorem 3. *For $n \geq 3$, $H_2(\mathrm{IA}_n^+, \mathbf{Z})$ contains a free abelian group of rank*

$$\frac{1}{72}n(n^2-1)(n-2)^2(n^2+5n+9) + \frac{1}{2}(n-1)(n-2).$$

By considering the dual version of the above argument, we also see that $H^2(\mathrm{IA}_n^+, \mathbf{Z})$ contains a free abelian group of rank $n(n^2-1)(n-2)^2(n^2+5n+9)/72 + (n-1)(n-2)/2$. In addition to this, we see that the cup product

$$\cup : \Lambda^2 H^1(\mathrm{IA}_n^+, \mathbf{Z}) \to H^2(\mathrm{IA}_n^+, \mathbf{Z})$$

is not surjective. Namely,

Theorem 4. *For any $n \geq 3$, $H^2(\mathrm{IA}_n^+, \mathbf{Z}) \neq \mathrm{Im}(\cup)$.*

Finally, we give some remarks related to our results. We show that the natural homomorphisms

$$\mathcal{A}'_{F_n}(k)^+/\mathcal{A}'_{F_n}(k+1)^+ \to \mathcal{A}'_{F_n}(k)/\mathcal{A}'_{F_n}(k+1)$$

induced from the inclusion map $\mathrm{IA}_n^+ \to \mathrm{IA}_n$ are injective for any $k \geq 1$. The group $\mathrm{P}\Sigma_n^+$ of IA_n^+ generated by K_{ij} for any $1 \leq j < i \leq n$ is called the upper-triangular McCool group. Let $\mathrm{P}\Sigma_n(1)^+ \supset \mathrm{P}\Sigma_n(2)^+ \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n^+$. Cohen, Pakianathan, Vershinin and Wu [9] completely determined the structure of the graded quotients $\mathrm{P}\Sigma_n(k)^+/\mathrm{P}\Sigma_n(k+1)^+$ for each $k \geq 1$. By using the Johnson homomorphisms, we show that the natural homomorphisms

$$\mathrm{P}\Sigma_n(k)^+/\mathrm{P}\Sigma_n(k+1)^+ \to \mathcal{A}'_{F_n}(k)/\mathcal{A}'_{F_n}(k+1)$$

induced from the inclusion map $\mathrm{P}\Sigma_n^+ \to \mathrm{IA}_n$ are injective for any $k \geq 1$. We remark that Part (2) of this proposition is the answer to a problem listed in [9]. (See Section 10 of [9].)

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